SPECHT MODULES AND KAZHDAN–LUSZTIG CELLS IN TYPE B_n

MEINOLF GECK, LACRIMIOARA IANCU AND CHRISTOS PALLIKAROS

Abstract. Dipper, James and Murphy generalized the classical Specht module theory to Hecke algebras of type B_n. On the other hand, for any choice of a monomial order on the parameters in type B_n, we obtain corresponding Kazhdan–Lusztig cell modules. In this paper, we show that the Specht modules are naturally isomorphic to the Kazhdan–Lusztig cell modules *if* we choose the dominance order on the parameters, as in the "asymptotic case" studied by Bonnafé and the second named author. We also give examples which show that such an isomorphism does not exist for other choices of monomial orders.

1. Introduction

Let $\mathcal{H}_n$ be the generic Iwahori–Hecke algebra of type A_{n-1} or B_n. For any partition or bipartition λ of n, we have a corresponding Specht module $\tilde{S}^\lambda$, as defined by Dipper–James [5] (in type A_{n-1}) and Dipper–James–Murphy [6] (in type B_n). On the other hand, we have the cell modules arising from the theory of Kazhdan–Lusztig cells; see Lusztig [13], [14]. Now McDonough–Pallikaros [15] showed that, in type A_{n-1}, the Specht modules and Kazhdan–Lusztig cell modules are naturally isomorphic. The main purpose of this paper is to prove an analogous result for type B_n. Note that, contrary to the situation in type A_{n-1}, there are many different types of Kazhdan–Lusztig cell modules in type B_n, depending on the choice of a monomial order on the two parameters in type B_n. We will show that it is precisely the "asymptotic case" studied in [3] which yields an isomorphism with the Specht modules of Dipper–James–Murphy.

In Theorem 3.6, we show the existence of a canonical isomorphism between a Specht module and a Kazhdan–Lusztig left cell module in the "asymptotic case" (where both of them are labelled by the appropriate bipartition of n). Both the Specht modules and the Kazhdan–Lusztig cells have certain standard bases. We show that, for a suitable ordering of these bases, the matrix of the canonical isomorphism is triangular with 1 on the diagonal. Our proof essentially relies on the combinatorial description [3] of the left cells in the "asymptotic case". This allows us to determine explicitly (in terms of reduced expressions of elements) certain distinguished left cells for every bipartition of n; see Proposition 2.6.

In Section 4, we give examples which show that the Specht modules are *not* isomorphic to Kazhdan–Lusztig cell modules for choices of the monomial order which are different from the "asymptotic case".

2. Kazhdan–Lusztig bases and cells

In this section, we recall the basic definitions concerning Kazhdan–Lusztig bases and cells, following Lusztig [13], [14]. We also recall some of the main results of [3], [4], [8] concerning the "asymptotic case" in type B_n. This will allow us, see Proposition 2.6, to describe explicit reduced expressions for the elements in certain distinguished left cells in type B_n.

Date: June, 2007.
2000 *Mathematics Subject Classification.* Primary 20C08; Secondary 20G40.

2.A. **Basic definitions.**

In [14], an Iwahori–Hecke algebra with possibly unequal parameters is defined with respect to an integer-valued weight function on W. Following a suggestion of Bonnafé [4], we can slightly modify Lusztig's definition so as to include the more general setting in [13] as well.

Let Γ be an abelian group (written additively) and let $A = \mathbb{Z}[\Gamma]$ be the free abelian group with basis $\{\varepsilon^\gamma \mid \gamma \in \Gamma\}$. There is a well-defined ring structure on A such that $\varepsilon^\gamma\varepsilon^{\gamma'} = \varepsilon^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. (Hence, if $\Gamma = \mathbb{Z}$, then A is nothing but the ring of Laurent polynomials in an indeterminate ε.) We write $1 = \varepsilon^0 \in A$. Given $a \in A$ we denote by a_γ the coefficient of ε^γ, so that $a = \sum_{\gamma\in\Gamma} a_\gamma\varepsilon^\gamma$. We say that a function

$$L\colon W \to \Gamma$$

is a weight function if $L(ww') = L(w) + L(w')$ whenever we have $\ell(ww') = \ell(w) + \ell(w')$ where $\ell\colon W \to \mathbb{N}$ is the usual length function. (We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$.) Let $\mathcal{H} = \mathcal{H}(W, S, L)$ be the generic Iwahori–Hecke algebra over A with parameters $\{v_s \mid s \in S\}$ where $v_s := \varepsilon^{L(s)}$ for $s \in S$. The algebra $\mathcal{H}$ is free over A with basis $\{T_w \mid w \in W\}$, and the multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where $s \in S$ and $w \in W$.

Now assume that there is a total order $\leqslant$ on Γ compatible with the group structure. (In the setting of [14], $\Gamma = \mathbb{Z}$ with the natural order.) The following definitions will depend on the choice of this total order. We denote by $A_{\geqslant 0}$ the set of $\mathbb{Z}$-linear combinations of elements ε^γ where $\gamma \geqslant 0$. Similarly, we define $A_{>0}$, $A_{\leqslant 0}$ and $A_{<0}$. We assume throughout that $L(s) > 0$ for all $s \in S$. Having fixed a total order on Γ, we have a corresponding Kazhdan–Lusztig basis $\{C_w \mid w \in W\}$ of $\mathcal{H}$. The element C_w is self-dual with respect to a certain ring involution of $\mathcal{H}$, and we have

$$C_w = T_w + \sum_{\substack{y\in W\\ y<w}} P^*_{y,w}\, T_y \in \mathcal{H},$$

where $<$ denotes the Bruhat–Chevalley order on W and $P^*_{y,w} \in A_{<0}$ for all $y < w$ in W; see [13, §6]. (In the framework of [14], the polynomials $P^*_{y,w}$ are denoted $p_{y,w}$ and the basis elements C_w are denoted c_w.) Given $x, y \in W$, we write

$$C_x\, C_y = \sum_{z\in W} h_{x,y,z}\, C_z \qquad \text{where } h_{x,y,z} \in A.$$

We have the following more explicit multiplication rules (see [13, §6]): for $w \in W$ and $s \in S$, we have

$$T_s\, C_w = \begin{cases} C_{sw} - v_s^{-1}C_w + \displaystyle\sum_{\substack{z<w\\ sz<z}} M^s_{z,w}\, C_z & \text{if } sw > w, \\ v_s\, C_w & \text{if } sw < w, \end{cases}$$

where the elements $M^s_{z,w} \in A$ are determined as in [13, §3].

We recall the definition of the left cells of W and the corresponding left cell representations of $\mathcal{H}$ (see [13] or [14]). Note again that these depend on the choice of a total order on Γ.

We write $z \leftarrow_{\mathcal{L}} y$ if there exists some $s \in S$ such that $h_{s,y,z} \neq 0$, that is, C_z occurs in $C_s\, C_y$ (when expressed in the C-basis). Let $\leqslant_{\mathcal{L}}$ be the pre-order relation on W generated by $\leftarrow_{\mathcal{L}}$, that is, we have $z \leqslant_{\mathcal{L}} y$ if there exist elements $z = z_0, z_1, \ldots, z_k = y$ such that $z_{i-1} \leftarrow_{\mathcal{L}} z_i$ for $1 \leqslant i \leqslant k$. The equivalence relation associated with $\leqslant_{\mathcal{L}}$ will be denoted by $\sim_{\mathcal{L}}$ and the corresponding equivalence classes are called the *left cells* of W.

Similarly, we can define a pre-order $\leqslant_{\mathcal{R}}$ by considering multiplication by C_s on the right in the defining relation. The equivalence relation associated with $\leqslant_{\mathcal{R}}$ will be denoted by $\sim_{\mathcal{R}}$ and the corresponding equivalence classes are called the *right cells* of W. We have

$$x \leqslant_{\mathcal{R}} y \quad \Leftrightarrow \quad x^{-1} \leqslant_{\mathcal{L}} y^{-1}.$$

This follows by using the anti-automorphism $\flat\colon \mathcal{H} \to \mathcal{H}$ given by $T_w^\flat = T_{w^{-1}}$; we have $C_w^\flat = C_{w^{-1}}$; see [14, 5.6]. Thus, any statement concerning the left pre-order relation $\leqslant_{\mathcal{L}}$ has an equivalent version for the right pre-order relation $\leqslant_{\mathcal{R}}$, via $\flat$.

Finally, we define a pre-order $\leqslant_{\mathcal{LR}}$ by the condition that $x \leqslant_{\mathcal{LR}} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leqslant_{\mathcal{L}} x_i$ or $x_{i-1} \leqslant_{\mathcal{R}} x_i$. The equivalence relation associated with $\leqslant_{\mathcal{LR}}$ will be denoted by $\sim_{\mathcal{LR}}$ and the corresponding equivalence classes are called the *two-sided cells* of W.

Each left cell $\mathfrak{C}$ gives rise to a representation of $\mathcal{H}$. This is constructed as follows (see [13, §7]). Let

$$\begin{aligned} \mathfrak{I}_{\mathfrak{C}} &= \langle C_y \ (y \in W) \mid y \leqslant_{\mathcal{L}} w \text{ for some } w \in \mathfrak{C} \rangle_A, \\ \hat{\mathfrak{I}}_{\mathfrak{C}} &= \langle C_y \ (y \in W) \mid y \leqslant_{\mathcal{L}} w \text{ for some } w \in \mathfrak{C} \text{ and } y \notin \mathfrak{C} \rangle_A. \end{aligned}$$

These are left ideals in $\mathcal{H}$. Hence $[\mathfrak{C}]_A = \mathfrak{I}_{\mathfrak{C}}/\hat{\mathfrak{I}}_{\mathfrak{C}}$ is a left $\mathcal{H}$-module; it is free over A with basis $\{e_w \mid w \in \mathfrak{C}\}$ where e_w denotes the class of C_w modulo $\hat{\mathfrak{I}}_{\mathfrak{C}}$. Explicitly, the action of $\mathcal{H}$ on $[\mathfrak{C}]_A$ is given by

$$C_w.e_x = \sum_{y \in \mathfrak{C}} h_{w,x,y}\, e_y \qquad \text{for all } x \in \mathfrak{C} \text{ and } w \in W.$$

2.B. **The "asymptotic case" in type B_n.**

Now let $\Gamma = \mathbb{Z}^2$; then $A = \mathbb{Z}[\Gamma]$ is nothing but the ring of Laurent polynomials in two independent indeterminates $V = \varepsilon^{(1,0)}$ and $v = \varepsilon^{(0,1)}$. Let $W = W_n$ be the Coxeter group of type B_n ($n \geqslant 2$), with generators, relations and weight function $L\colon W_n \to \Gamma$ given by the following diagram:

$$\begin{array}{lcccccc} & t & & s_1 & & s_2 & & & s_{n-1} \\ B_n & \circ & = & \circ & - & \circ & - & \cdots & - & \circ \\ L: & b & & a & & a & & & a \end{array}$$

where $a, b \in \Gamma$. Let $\mathcal{H}_n$ be the corresponding generic two-parameter Iwahori–Hecke algebra over $A = \mathbb{Z}[\Gamma]$, where we set

$$V := v_t = \varepsilon^b \qquad \text{and} \qquad v := v_{s_1} = \cdots = v_{s_{n-1}} = \varepsilon^a.$$

(Note that *any* Hecke algebra of type B_n can be obtained from $\mathcal{H}_n$ by "specialisation"; see also Remark 3.8 below.) In order to obtain Kazhdan–Lusztig cells and the corresponding cell modules, we have to specify a total order $\leqslant$ on Γ. Note that there are infinitely many such total orders: For example, we have all the weighted lexicographic orders, given by $(i, j) < (i', j')$ if and only if $xi + yj < xi' + yj'$ or $xi + yj = xi' + yj'$ and $i < i'$, where x, y are fixed positive real numbers.

Here, we shall take for $\leqslant$ the lexicographic order on Γ such that

$$\boxed{(i, j) < (i', j') \quad \overset{\text{def}}{\Longleftrightarrow} \quad i < i' \quad \text{or} \quad i = i' \text{ and } j < j'.}$$

This is the set-up originally considered by Bonnafé–Iancu [3]; it is called the *"asymptotic case"* in type B_n. We shall need some notation from [3]. Given $w \in W_n$, we denote by $\ell_t(w)$ the number of occurrences of the generator t in a reduced expression for w, and call this the "t-length" of w.

The parabolic subgroup $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$ is naturally isomorphic to the symmetric group on $\{1, \ldots, n\}$, where s_i corresponds to the basic transposition $(i, i+1)$. For $1 \leqslant l \leqslant n-1$, we set $\Sigma_{l,n-l} := \{s_1, \ldots, s_{n-1}\} \setminus \{s_l\}$. For $l = 0$ or $l = n$, we also set $\Sigma_{0,n} = \Sigma_{n,0} = \{s_1, \ldots, s_{n-1}\}$. Then we have the Young subgroup

$$\mathfrak{S}_{l,n-l} = \langle \Sigma_{l,n-l}\rangle = \mathfrak{S}_{\{1,\ldots,l\}} \times \mathfrak{S}_{\{l+1,\ldots,n\}}.$$

Let $Y_{l,n-l}$ be the set of distinguished left coset representatives of $\mathfrak{S}_{l,n-l}$ in $\mathfrak{S}_n$. We have the parabolic subalgebra $\mathcal{H}_{l,n-l} = \langle T_\sigma \mid \sigma \in \mathfrak{S}_{l,n-l}\rangle_A \subseteq \mathcal{H}_n$.

We denote by $\leqslant_{\mathcal{L},l}$ the Kazhdan–Lusztig (left) pre-order relation on $\mathfrak{S}_{l,n-l}$ and by $\sim_{\mathcal{L},l}$ the corresponding equivalence relation. The symbols $\leqslant_{\mathcal{R},l}$, $\leqslant_{\mathcal{LR},l}$, $\sim_{\mathcal{R},l}$ and $\sim_{\mathcal{LR},l}$ have a similar meaning.

Furthermore, as in [3, §4], we set $a_0 = 1$ and

$$a_l := t(s_1t)(s_2s_1t)\cdots(s_{l-1}s_{l-2}\cdots s_1t) \qquad \text{for } l > 0.$$

Then, by [3, Prop. 4.4], the set $Y_{l,n-l}a_l$ is precisely the set of distinguished left coset representatives of $\mathfrak{S}_n$ in W_n whose t-length equals l. Furthermore, every element $w \in W_n$ has a unique decomposition

$$w = a_w a_l \sigma_w b_w^{-1} \qquad \text{where } l = \ell_t(w),\ \sigma_w \in \mathfrak{S}_{l,n-l} \text{ and } a_w, b_w \in Y_{l,n-l};$$

see [3, 4.6]. We call this the *Clifford normal form* of w.

Theorem 2.1 (Bonnafé–Iancu [3, §7])**.** *Assume that we are in the "asymptotic case" defined above. Let $x, y \in W_n$. Then $x \sim_{\mathcal{L}} y$ if and only if $l := \ell_t(x) = \ell_t(y)$, $b_x = b_y$ and $\sigma_x \sim_{\mathcal{L},l} \sigma_y$.*

Example 2.2. Let $l \in \{0, \ldots, n\}$ and $\mathfrak{C}$ be a left cell of $\mathfrak{S}_{l,n-l}$. Since this group is a direct product, we can write $\mathfrak{C} = \mathfrak{C}_1 \cdot \mathfrak{C}_2$ where $\mathfrak{C}_1$ is a left cell in $\mathfrak{S}_{\{1,\ldots,l\}}$ and $\mathfrak{C}_2$ is a left cell in $\mathfrak{S}_{\{l+1,\ldots,n\}}$. Now Theorem 2.1 implies that

(a) $\qquad Y_{l,n-l}\, a_l\, \mathfrak{C}$ is a left cell of W_n (in the "asymptotic case").

Now recall from [3, 4.1] that $a_l = w_l\sigma_l$ where w_l is the longest element of the parabolic subgroup $W_l = \langle t, s_1, \ldots, s_{l-1}\rangle$ (of type B_l) and σ_l is the longest element of $\mathfrak{S}_{\{1,\ldots,l\}}$. Since w_l is central in W_l and conjugation with σ_l preserves the left cells of $\mathfrak{S}_{\{1,\ldots,l\}}$, we conclude that $a_l\mathfrak{C}_1a_l$ is a left cell of $\mathfrak{S}_{\{1,\ldots,l\}}$, too. Furthermore, a_l commutes with all elements of $\mathfrak{S}_{\{l+1,\ldots,n\}}$ and so $a_l\mathfrak{C}a_l$ is a left cell of $\mathfrak{S}_{l,n-l}$. Applying (a) now yields that

(b) $\qquad Y_{l,n-l}\, \mathfrak{C}\, a_l$ is a left cell of W_n (in the "asymptotic case").

This example will be useful in the proof of Proposition 2.6 below.

2.C. **Bitableaux.**

Let Λ_n be the set of all bipartitions of n. We write such a bipartition in the form $\lambda = (\lambda_1|\lambda_2)$ where λ_1 and λ_2 are partitions such that $|\lambda_1| + |\lambda_2| = n$. For $\lambda \in \Lambda_n$, let $\mathbb{T}(\lambda)$ be the set of all standard λ-bitableaux. (Whenever we speak of bitableaux, it is understood that the filling is by the numbers $1, \ldots, n$.) The generalized Robinson–Schensted correspondence of [3] is a bijection

$$W_n \xrightarrow{\sim} \coprod_{\lambda \in \Lambda_n} \mathbb{T}(\lambda) \times \mathbb{T}(\lambda), \qquad w \mapsto (P(w), Q(w)).$$

Thus, to each element $w \in W_n$, we associate a pair of λ-bitableux for some $\lambda \in \Lambda_n$; in this case, we also write $w \rightsquigarrow \lambda$ and say that w is type λ.

The following result provides an explicit combinatorial description of the left, right and two-sided cells in the "asymptotic case" in type B_n.

Theorem 2.3. *Assume we are in the "asymptotic case" defined in §2.B. Let $x, y \in W_n$.*

(a) (Bonnafé–Iancu [3, §7]) *We have $x \sim_{\mathcal{L}} y$ if and only if $Q(x) = Q(y)$. Furthermore, $x \sim_{\mathcal{R}} y$ if and only if $P(x) = P(y)$.*
(b) (Bonnafé [4, §3]) *We have $x \sim_{\mathcal{LR}} y$ if and only if all of $P(x)$, $P(y)$, $Q(x)$ and $Q(y)$ have the same shape.*

Now let $\mathfrak{C}$ be a left cell of W_n. We shall say that $\mathfrak{C}$ is of type $\lambda \in \Lambda_n$ if the bitableaux $Q(x)$ (where $x \in \mathfrak{C}$) have shape λ.

Theorem 2.4 (Geck [8, Theorem 6.3]). *Let $\mathfrak{C}$ and $\mathfrak{C}'$ be left cells of W_n (in the "asymptotic case") which have the same type $\lambda \in \Lambda_n$. Then the left cell modules $[\mathfrak{C}]_A$ and $[\mathfrak{C}']_A$ are canonically isomorphic. In fact, there is a bijection $\mathfrak{C} \leftrightarrow \mathfrak{C}'$ which induces an $\mathcal{H}_n$-module isomorphism $[\mathfrak{C}]_A \xrightarrow{\sim} [\mathfrak{C}']_A$.*

The above results show that, in order to study the left cell modules of $\mathcal{H}_n$, it is sufficient to exhibit one particular left cell of type λ, for each given $\lambda \in \Lambda_n$. For this purpose, we shall need some further combinatorial notions from Dipper–James–Murphy [6, §3].

So let us fix a bipartition $\lambda = (\lambda_1|\lambda_2) \in \Lambda_n$, where $l = |\lambda_1|$ and $0 \leqslant l \leqslant n$. Let $\mathfrak{t}^\lambda$ be the "canonical" standard bitableau of shape λ defined in [6, p. 508]. Thus, $\mathfrak{t}^\lambda$ is a pair consisting of the "canonical" λ_1-tableau $\mathfrak{t}^{\lambda_1}$ (obtaining by filling the rows in order from left to right by the numbers $1, \ldots, l$) and the "canonical" λ_2-tableau $\mathfrak{t}'^{\lambda_2}$ (obtained by filling the rows in order from left to right by the numbers $l+1, \ldots, n$).

The symmetric group $\mathfrak{S}_n$ acts (on the left) on bitableaux by permuting the entries. If $\mathfrak{t}$ is any bitableau of shape λ, denote by $d(\mathfrak{t})$ the unique element of $\mathfrak{S}_n$ which sends $\mathfrak{t}^\lambda$ to $\mathfrak{t}$. Thus, we have $d(\mathfrak{t}).\mathfrak{t}^\lambda = \mathfrak{t}$ for any λ-bitableau $\mathfrak{t}$. Now let $\mathbb{T}^r(\lambda)$ denote the set of all row-standard λ-bitableaux. Then

$$Y^\lambda := \{d(\mathfrak{t}) \mid \mathfrak{t} \in \mathbb{T}^r(\lambda)\}$$

is the set of distinguished left coset representatives of the parabolic subgroup $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$; see [6, p. 509]. Applying this to the bipartition $((l),(n-l))$, we find that

$$Y_{l,n-l} = Y^{((l),(n-l))}.$$

Now we also define $\mathbb{T}^r_l(\lambda)$ to be the set of all $\mathfrak{t} = (\mathfrak{t}_1|\mathfrak{t}_2) \in \mathbb{T}^r(\lambda)$ where $\mathfrak{t}_1$ is filled by the numbers $1, \ldots, l$ and $\mathfrak{t}_2$ is filled by the numbers $l+1, \ldots, n$. Then, by the same argument as above,

$$Y^\lambda_l := \{d(\mathfrak{t}) \mid \mathfrak{t} \in \mathbb{T}^r_l(\lambda)\}$$

is the set of distinguished left coset representatives of the parabolic subgroup $\mathfrak{S}_\lambda$ inside $\mathfrak{S}_{l,n-l}$. Hence, considering the chain of parabolic subgroups $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_{l,n-l} \subseteq \mathfrak{S}_n$, we obtain a decomposition

$$Y^\lambda = Y_{l,n-l} \cdot Y^\lambda_l$$

where $\ell(yd(\mathfrak{t})) = \ell(y) + \ell(d(\mathfrak{t}))$ for all $y \in Y_{l,n-l}$ and $\mathfrak{t} \in \mathbb{T}^r_l(\lambda)$.

Now we have the following purely combinatorial result.

Lemma 2.5. *In the above setting, let $\mathfrak{s} \in \mathbb{T}^r(\lambda)$, $\mathfrak{t} \in \mathbb{T}^r_l(\lambda)$ and $y \in Y_{l,n-l}$ be such that $d(\mathfrak{s}) = y\,d(\mathfrak{t})$. Then $\mathfrak{s}$ is a standard bitableau if and only if $\mathfrak{t}$ is a standard bitableau.*

Proof. We have $\mathfrak{s} = d(\mathfrak{s}).\mathfrak{t}^\lambda = (yd(\mathfrak{t})).\mathfrak{t}^\lambda = y.(d(\mathfrak{t}).\mathfrak{t}^\lambda) = y.\mathfrak{t}$. The permutation $y \in Y_{l,n-l}$ has the property that $y(i) < y(i+1)$ for $1 \leqslant i < l$ and $y(i) < y(i+1)$ for $l \leqslant i < n$. Now it is an easy combinatorial exercise to see that $\mathfrak{s}$ is standard if and only if $\mathfrak{t}$ is standard; we omit further details. $\square$

Proposition 2.6. *Let $\lambda = (\lambda_1|\lambda_2) \in \Lambda_n$ and $l = |\lambda_1|$. Let $\sigma_\lambda \in \mathfrak{S}_\lambda$ be the longest element and $\mathfrak{C}_\lambda$ be the left cell (with respect to the "asymptotic case") containing $\sigma_\lambda a_l \in W_n$. Then $\mathfrak{C}_\lambda$ has type $(\lambda_2^*|\lambda_1)$ and we have*

$$\mathfrak{C}_\lambda = \{d(\mathfrak{t})\,\sigma_\lambda\, a_l \mid \mathfrak{t} \in \mathbb{T}(\lambda)\},$$

where $\ell(d(\mathfrak{t})\sigma_\lambda a_l) = \ell(d(\mathfrak{t})) + \ell(\sigma_\lambda a_l)$ for all $\mathfrak{t} \in \mathbb{T}(\lambda)$.

Proof. By relation (♠) in the proof of [10, Prop. 5.4], the element $a_l\sigma_\lambda$ has type $(\lambda_2^*|\lambda_1)$. Now since $\sigma_\lambda a_l = (a_l\sigma_\lambda)^{-1}$ it follows that $\sigma_\lambda a_l$ also has type $(\lambda_2^*|\lambda_1)$. Now, by [15, Lemma 3.3] (extended to the direct product of two symmetric groups), the set

$$\mathfrak{C} := \{d(\mathfrak{t})\,\sigma_\lambda \mid \mathfrak{t} \in \mathbb{T}_l(\lambda)\}$$

is the left cell of $\mathfrak{S}_{l,n-l}$ containing σ_λ, where $\mathbb{T}_l(\lambda)$ is the set of all standard λ-bitableaux in $\mathbb{T}_l^r(\lambda)$. Hence, by Example 2.2(b), we have

$$\mathfrak{C}_\lambda = \{y\,d(\mathfrak{t})\,\sigma_\lambda\, a_l \mid y \in Y_{l,n-l},\, \mathfrak{t} \in \mathbb{T}_l(\lambda)\}.$$

Furthermore, $\ell(y\,d(\mathfrak{t})\,\sigma_\lambda\, a_l) = \ell(y\,d(\mathfrak{t})) + \ell(\sigma_\lambda\, a_l)$. Now it remains to use Lemma 2.5. □

Remark 2.7. In the above setting, it is not difficult to prove the following related result. Let $x \in W_n$ and $l := \ell_t(x)$. Then we have:

$$x \leqslant_{\mathcal{L}} \sigma_\lambda a_l \iff x = d(\mathfrak{s})\sigma_\lambda a_l \text{ where } \mathfrak{s} \text{ is a row-standard } \lambda\text{-bitableaux.}$$

This follows from the properties of the Clifford normal form of the elements in W_n established in [3, §7] and the refinement obtained in [8, Theorem 5.11]. As we do not need this result in this paper, we omit further details.

3. Specht modules

We keep the setup of the previous section, where we consider the Iwahori–Hecke algebra $\mathcal{H}_n$ of type B_n, defined over a polynomial ring $A = \mathbb{Z}[V^{\pm 1}, v^{\pm 1}]$ in two independent indeterminates. We now consider the Specht modules defined by Dipper–James–Murphy [6]. The definition is based on the construction of a new basis of $\mathcal{H}_n$, which is of the form

$$\{x_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}\, x_\lambda\, T_{d(\mathfrak{t})^{-1}} \mid \lambda \in \Lambda_n \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)\}$$

where the element x_λ is defined in [6, 4.1]; note that the definition of x_λ does not rely on the choice of a total order on Γ. (An explicit description of x_λ will be given in Lemma 3.2 below.)

Let $N^\lambda \subseteq \mathcal{H}_n$ be the A-submodule spanned by all $x_{\mathfrak{s}\mathfrak{t}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are standard μ-bitableaux such that $\lambda \trianglelefteq \mu$. Here, $\trianglelefteq$ denotes the dominance order on bipartitions, which is defined as follows; see Dipper–James–Murphy [6, §3]: Let $\lambda = (\lambda_1|\lambda_2)$ and $\mu = (\mu_1|\mu_2)$ be bipartitions of n, with parts

$$\lambda_1 = (\lambda_1^{(1)} \geqslant \lambda_1^{(2)} \geqslant \cdots \geqslant 0), \qquad \lambda_2 = (\lambda_2^{(1)} \geqslant \lambda_2^{(2)} \geqslant \cdots \geqslant 0),$$
$$\mu_1 = (\mu_1^{(1)} \geqslant \mu_1^{(2)} \geqslant \cdots \geqslant 0), \qquad \mu_2 = (\mu_2^{(1)} \geqslant \mu_2^{(2)} \geqslant \cdots \geqslant 0).$$

Then $\lambda \trianglelefteq \mu$ if

$$\sum_{i=1}^{j} \lambda_1^{(i)} \leqslant \sum_{i=1}^{j} \mu_1^{(i)} \quad (\forall j) \quad \text{and} \quad |\lambda_1| + \sum_{i=1}^{j} \lambda_2^{(i)} \leqslant |\mu_1| + \sum_{i=1}^{j} \mu_2^{(i)} \quad (\forall j).$$

By [6, Cor. 4.13], N^λ is a two-sided ideal of $\mathcal{H}_n$. Since the basis elements T_w ($w \in W_n$) are invertible in $\mathcal{H}_n$, we conclude that

$$N^\lambda = \sum_{\mu \in \Lambda_n;\, \lambda \trianglelefteq \mu} \mathcal{H}_n x_\mu \mathcal{H}_n.$$

Similarly, we have the two-sided ideal $\hat{N}^\lambda$ spanned by all $x_{\mathfrak{st}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are standard μ-bitableaux such that $\lambda \lhd \mu$ (that is, $\lambda \unlhd \mu$ but $\lambda \neq \mu$).

Definition 3.1 (Dipper–James–Murphy [6, Def. 4.19])**.** Let $\lambda \in \Lambda_n$. The corresponding *Specht module* is defined by

$$\tilde{S}^\lambda := M^\lambda/(M^\lambda \cap \hat{N}^\lambda) \qquad \text{where} \qquad M^\lambda = \mathcal{H}_n x_\lambda.$$

By [6, Theorem 4.20], $\tilde{S}^\lambda$ is free over A, with standard basis $\{x_\mathfrak{s} \mid \mathfrak{s} \in \mathbb{T}(\lambda)\}$ where $x_\mathfrak{s}$ denotes the class modulo $M^\lambda \cap \hat{N}^\lambda$ of the element $x_{\mathfrak{s}\mathfrak{t}^\lambda} \in M^\lambda$.

Our task will be to identify these Specht modules with certain Kazhdan–Lusztig left cells modules. For this purpose, assume from now on that we have chosen a total order on Γ such that we are in the "asymptotic case" defined in §2.B. Our first result, which is based on Bonnafé [4], identifies x_λ in terms of the corresponding Kazhdan–Lusztig basis of $\mathcal{H}_n$.

Lemma 3.2. *Let $\lambda = (\lambda_1|\lambda_2) \in \Lambda_n$ and $l = |\lambda_1|$. Then*

$$V^l v^{l(l-1)-\ell(\sigma_\lambda)}\, x_\lambda = T_{\sigma_l}\, C_{a_l\sigma_\lambda} = C_{\sigma_\lambda a_l} T_{\sigma_l},$$

where the elements σ_l, a_l and σ_λ are defined in §2.

Proof. In [6, 4.1], the element x_λ is defined as the product of three commuting factors u_l^+, x_{λ_1}, x_{λ_2}. Bonnafé's formula [4, Prop. 2.5] shows that

$$V^l v^{l(l-1)}\, u_l^+ = C_{a_l} T_{\sigma_l} = T_{\sigma_l} C_{a_l}\,.$$

Furthermore, by Lusztig [14, Cor. 12.2], we have $x_{\lambda_1} x_{\lambda_2} = v^{\ell(\sigma_\lambda)} C_{\sigma_\lambda}$. Finally, by [4, Prop. 2.3], we have $C_{a_l} C_{\sigma_\lambda} = C_{a_l\sigma_\lambda}$ and $C_{\sigma_\lambda} C_{a_l} = C_{\sigma_\lambda a_l}$. This yields the desired formulas. □

Corollary 3.3. *Let $\lambda \in \Lambda_n$. Then $M^\lambda = \mathcal{H}_n\, C_{a_l\sigma_\lambda} = \mathcal{H}_n C_{\sigma_\lambda a_l} T_{\sigma_l}$.*

Proof. Clear by Lemma 3.2; just note v, V and T_{σ_l} are invertible in $\mathcal{H}_n$. □

Proposition 3.4. *Let $\lambda \in \Lambda_n$. Then we have*

(a)
$$\begin{aligned} N^\lambda &= \langle C_y\ (y \in W_n) \mid y \rightsquigarrow (\nu_1|\nu_2) \text{ where } (\lambda_1|\lambda_2) \unlhd (\nu_2|\nu_1^*)\rangle_A \\ &\supseteq \langle C_y\ (y \in W_n) \mid y \leqslant_{\mathcal{LR}} a_l\sigma_\lambda\rangle_A, \end{aligned}$$

(b)
$$\hat{N}^\lambda = \langle C_y\ (y \in W_n) \mid C_y \in N^\lambda \text{ and } y \not\sim_{\mathcal{LR}} a_l\sigma_\lambda\rangle_A.$$

Proof. (a) The equality is proved in [10, Theorem 1.5]. Now let $y \in W_n$ be such that $y \leqslant_{\mathcal{LR}} a_l\sigma_\lambda$. Assume that $y \rightsquigarrow (\mu_1|\mu_2)$. Then Proposition 2.6 and [10, Prop. 5.4] show that $(\mu_1|\mu_2^*) \unlhd (\lambda_2^*|\lambda_1^*)$ or, equivalently, $(\lambda_1|\lambda_2) \unlhd (\mu_2|\mu_1^*)$. Thus, we have $C_y \in N^\lambda$, as required.

(b) Since $\hat{N}^\lambda$ is the sum of all N^μ where $\mu \in \Lambda_n$ and $\lambda \lhd \mu$, the equality in (a) also implies that

$$\hat{N}^\lambda = \langle C_y\ (y \in W_n) \mid y \rightsquigarrow (\nu_1|\nu_2) \text{ where } (\lambda_1|\lambda_2) \lhd (\nu_2|\nu_1^*)\rangle_A.$$

So (b) follows from the description of the two-sided cells in Theorem 2.3(b). □

Now we are ready to construct a canonical homomorphism from a Specht module to a certain Kazhdan–Lusztig cell module.

Lemma 3.5. *Let $\lambda = (\lambda_1|\lambda_2) \in \Lambda_n$ and $l = |\lambda_1|$. Let $\mathfrak{C}_\lambda$ be the left cell of W_n containing $\sigma_\lambda a_l$ (with respect to the "asymptotic case"); see Proposition 2.6. Then there is a unique $\mathcal{H}_n$-module homomorphism $\varphi_\lambda\colon \tilde{S}^\lambda \to [\mathfrak{C}_\lambda]_A$ which sends the class of $x_\lambda \in M^\lambda$ in $\tilde{S}^\lambda$ to the class of $C_{\sigma_\lambda a_l} \in \mathfrak{I}_\lambda$ in $[\mathfrak{C}_\lambda]_A$.*

Proof. Recall that $[\mathfrak{C}_\lambda]_A = \mathfrak{I}_\lambda/\hat{\mathfrak{I}}_\lambda$, where

$$\mathfrak{I}_\lambda = \langle C_y \mid y \in W_n \text{ such that } y \leqslant_{\mathcal{L}} \sigma_\lambda a_l \rangle_A,$$
$$\hat{\mathfrak{I}}_\lambda = \langle C_y \mid y \in W_n \text{ such that } y \leqslant_{\mathcal{L}} \sigma_\lambda a_l \text{ and } y \notin \mathfrak{C}_\lambda \rangle_A.$$

We define $\zeta_\lambda := V^{-(l-1)}\, v^{\ell(\sigma_\lambda)-l(l-1)}\, T_{\sigma_l}^{-1} \in \mathcal{H}_n$. (Note that any element of the T-basis is invertible in $\mathcal{H}_n$.) Then the map

$$\rho_\lambda\colon \mathcal{H}_n \to \mathcal{H}_n, \qquad h \mapsto h\,\zeta_\lambda,$$

(that is, right multiplication by ζ_λ) is a left $\mathcal{H}_n$-module isomorphism. By Lemma 3.2, Corollary 3.3 and the definition of $\leqslant_{\mathcal{L}}$, we have

$$\rho_\lambda(x_\lambda) = C_{\sigma_\lambda a_l} \qquad \text{and} \qquad \rho_\lambda(M^\lambda) = \mathcal{H}_n\, C_{\sigma_\lambda a_l} \subseteq \mathfrak{I}_\lambda.$$

Now, by Proposition 3.4, we certainly have $\mathfrak{I}_\lambda \cap \hat{N}^\lambda \subseteq \hat{\mathfrak{I}}^\lambda$ and so

$$\rho_\lambda(M^\lambda \cap \hat{N}^\lambda) \subseteq \mathcal{H}_n C_{\sigma_\lambda a_l} \cap \hat{N}^\lambda \subseteq \mathfrak{I}_\lambda \cap \hat{N}^\lambda \subseteq \hat{\mathfrak{I}}_\lambda.$$

Hence, recalling also that $\tilde{S}^\lambda = M^\lambda/M^\lambda \cap \hat{N}^\lambda$, we obtain a well-defined $\mathcal{H}_n$-module homomorphism

$$\varphi_\lambda\colon \tilde{S}^\lambda \to [\mathfrak{C}_\lambda]_A, \qquad m + (M^\lambda \cap \hat{N}^\lambda) \mapsto m\zeta_\lambda + \hat{\mathfrak{I}}_\lambda,$$

having the desired properties. The unicity of φ_λ is clear since $\tilde{S}^\lambda$ is generated, as an $\mathcal{H}_n$-module, by the class of x_λ. □

Next, we would like to obtain more detailed information about the matrix of $\varphi_\lambda\colon \tilde{S}^\lambda \to [\mathfrak{C}_\lambda]_A$ with respect to the standard bases of the two modules. The aim will be to show that this matrix is triangular with 1 on the diagonal; in particular, this will show that φ_λ is an isomorphism.

Recall that the Specht module $\tilde{S}^\lambda$ has a standard basis $\{x_\mathfrak{s} \mid \mathfrak{s} \in \mathbb{T}(\lambda)\}$; see Definition 3.1. On the other hand, by the definition of cell modules and Proposition 2.6, $[\mathfrak{C}_\lambda]_A$ has a standard basis $\{e_{d(\mathfrak{s})\sigma_\lambda a_l} \mid \mathfrak{s} \in \mathbb{T}(\lambda)\}$ where $e_{d(\mathfrak{s})\sigma_\lambda a_l}$ denotes the class modulo $\hat{\mathfrak{I}}_\lambda$ of the element $C_{d(\mathfrak{s})\sigma_\lambda a_l} \in \mathfrak{I}_\lambda$. So, for any $\mathfrak{t} \in \mathbb{T}(\lambda)$, we write

$$\varphi_\lambda(x_\mathfrak{t}) = \sum_{\mathfrak{s} \in \mathbb{T}(\lambda)} g_{\mathfrak{s},\mathfrak{t}}\, e_{d(\mathfrak{s})\sigma_\lambda a_l} \quad \text{where} \quad g_{\mathfrak{s},\mathfrak{t}} \in A.$$

Thus, $G_\lambda := \big(g_{\mathfrak{s},\mathfrak{t}}\big)_{\mathfrak{s},\mathfrak{t}\in\mathbb{T}(\lambda)}$ is the matrix of φ_λ with respect to the standard bases of $\tilde{S}^\lambda$ and $[\mathfrak{C}_\lambda]_A$, respectively. Now we can state the main result of this paper.

Theorem 3.6. *The map $\varphi_\lambda\colon \tilde{S}^\lambda \to [\mathfrak{C}_\lambda]_A$ constructed in Lemma 3.5 is an isomorphism. More precisely, the following hold. For any $\mathfrak{s}, \mathfrak{t} \in \mathbb{T}(\lambda)$, we have*

$$g_{\mathfrak{t},\mathfrak{t}} = 1 \qquad \textit{for all } \mathfrak{t} \in \mathbb{T}(\lambda),$$
$$g_{\mathfrak{s},\mathfrak{t}} = 0 \qquad \textit{unless } d(\mathfrak{s}) \leqslant d(\mathfrak{t}),$$
$$g_{\mathfrak{s},\mathfrak{t}} \in v^{-1}\mathbb{Z}[v^{-1}] \qquad \textit{if } \mathfrak{s} \neq \mathfrak{t};$$

here, $\leqslant$ denotes the Bruhat–Chevalley order. Thus, the matrix G_λ has an upper unitriangular shape for a suitable ordering of the set $\mathbb{T}(\lambda)$.

Proof. We begin with the following computation inside the parabolic subgroup $\mathfrak{S}_n \subseteq W_n$. Let $\mathfrak{t} \in \mathbb{T}^r(\lambda)$. By the multiplication rules for the Kazhdan–Lusztig basis, $T_{d(\mathfrak{t})}C_{\sigma_\lambda}$ equals $C_{d(\mathfrak{t})\sigma_\lambda}$ plus a $\mathbb{Z}[v, v^{-1}]$-linear combination of terms C_x where $x \in \mathfrak{S}_n$, $x \leqslant_{\mathcal{L},n} \sigma_\lambda$ and $x < d(\mathfrak{t})\sigma_\lambda$. Now, the condition $x \leqslant_{\mathcal{L},n} \sigma_\lambda$ implies that x can be written as $x = d(\mathfrak{s})\sigma_\lambda$ for some $\mathfrak{s} \in \mathbb{T}^r(\lambda)$ (see, for

example, [15, 2.9]). Then the condition $x = d(\mathfrak{s})\sigma_\lambda < d(\mathfrak{t})\sigma_\lambda$ implies that $d(\mathfrak{s}) < d(\mathfrak{t})$ (see [14, 9.10(f)]). Thus, we obtain

$$(*)\qquad T_{d(\mathfrak{t})}\, C_{\sigma_\lambda} = \sum_{\mathfrak{s}\in\mathbb{T}^r(\lambda)} g'_{\mathfrak{s},\mathfrak{t}}\, C_{d(\mathfrak{s})\sigma_\lambda} \qquad \text{for any } \mathfrak{t}\in\mathbb{T}^r(\lambda),$$

where $g'_{\mathfrak{s},\mathfrak{t}}\in\mathbb{Z}[v,v^{-1}]$ for all $\mathfrak{s},\mathfrak{t}\in\mathbb{T}^r(\lambda)$; furthermore, $g'_{\mathfrak{t},\mathfrak{t}}=1$ and $g'_{\mathfrak{s},\mathfrak{t}}=0$ unless $d(\mathfrak{s})\leqslant d(\mathfrak{t})$ and $d(\mathfrak{s})\sigma_\lambda \leqslant_{\mathcal{L},n} d(\mathfrak{t})\sigma_\lambda$.

To pass from $\mathfrak{S}_n$ to W_n, we use the following argument. First note that a_l is a distinguished right coset representative of $\mathfrak{S}_n$ in W_n. By [4, Prop. 2.3], we have $C_{\sigma a_l} = C_\sigma C_{a_l}$ for any $\sigma\in\mathfrak{S}_n$. Hence, multiplying $(*)$ on the right by C_{a_l}, we obtain

$$T_{d(\mathfrak{t})}\, C_{\sigma_\lambda a_l} = \sum_{\mathfrak{s}\in\mathbb{T}^r(\lambda)} g'_{\mathfrak{s},\mathfrak{t}}\, C_{d(\mathfrak{s})\sigma_\lambda a_l} \qquad \text{for any } \mathfrak{t}\in\mathbb{T}^r(\lambda).$$

Now assume that $\mathfrak{t}\in\mathbb{T}(\lambda)$. Let $s\in\mathbb{T}^r(\lambda)$ be such that $g'_{\mathfrak{s},\mathfrak{t}}\neq 0$. Then $d(\mathfrak{s})\sigma_\lambda\leqslant_{\mathcal{L},n} d(\mathfrak{t})\sigma_\lambda$ and so $d(\mathfrak{s})\sigma_\lambda a_l \leqslant_{\mathcal{L}} d(\mathfrak{t})\sigma_\lambda a_l$; see [14, Prop. 9.11]. Hence, using Proposition 2.6, we find that

$$T_{d(\mathfrak{t})}\, C_{\sigma_\lambda a_l} \equiv \sum_{\mathfrak{s}\in\mathbb{T}(\lambda)} g'_{\mathfrak{s},\mathfrak{t}}\, C_{d(\mathfrak{s})\sigma_\lambda a_l} \quad \mod \quad \hat{\mathfrak{I}}_\lambda.$$

Passing to the quotient $\mathfrak{I}_\lambda \to [\mathfrak{C}_\lambda]_A = \mathfrak{I}_\lambda/\hat{\mathfrak{I}}_\lambda$, we obtain

$$T_{d(\mathfrak{t})}.e_{\sigma_\lambda a_l} = \sum_{\mathfrak{s}\in\mathbb{T}(\lambda)} g'_{\mathfrak{s},\mathfrak{t}}\, e_{d(\mathfrak{s})\,\sigma_\lambda a_l} \qquad \text{for any } \mathfrak{t}\in\mathbb{T}(\lambda).$$

Now note that $\varphi_\lambda(x_\mathfrak{t}) = \varphi_\lambda(T_{d(\mathfrak{t})}.\bar{x}_\lambda) = T_{d(\mathfrak{t})}.\varphi_\lambda(\bar{x}_\lambda) = T_{d(\mathfrak{t})}.e_{\sigma_\lambda a_l}$, where $\bar{x}_\lambda$ denotes the class of x_λ in $\tilde{S}^\lambda$. Thus, we see that $g'_{\mathfrak{s},\mathfrak{t}} = g_{\mathfrak{s},\mathfrak{t}}$ for all $\mathfrak{s},\mathfrak{t}\in\mathbb{T}(\lambda)$. Consequently, the coefficients $g_{\mathfrak{s},\mathfrak{t}}$ have the property that $g_{\mathfrak{t},\mathfrak{t}}=1$ and $g_{\mathfrak{s},\mathfrak{t}}=0$ unless $d(\mathfrak{s})\leqslant d(\mathfrak{t})$. Hence, for a suitable ordering of the rows and columns, the matrix G_λ is unitriangular and φ_λ is an isomorphism.

It remains to prove that $g_{\mathfrak{s},\mathfrak{t}}\in v^{-1}\mathbb{Z}[v^{-1}]$ for $\mathfrak{s}\neq\mathfrak{t}$. We will actually show that $g'_{\mathfrak{s},\mathfrak{t}}\in v^{-1}\mathbb{Z}[v^{-1}]$ for all $\mathfrak{s},\mathfrak{t}\in\mathbb{T}^r(\lambda)$ such that $\mathfrak{s}\neq\mathfrak{t}$. This is seen as follows. We can invert the equations $(*)$ and obtain

$$C_{d(\mathfrak{t})\,\sigma_\lambda} = \sum_{\mathfrak{s}\in\mathbb{T}^r(\lambda)} \tilde{g}_{\mathfrak{s},\mathfrak{t}}\, T_{d(\mathfrak{s})}\, C_{\sigma_\lambda} \qquad \text{for any } \mathfrak{t}\in\mathbb{T}^r(\lambda),$$

where the $\tilde{g}_{\mathfrak{s},\mathfrak{t}}$'s are the entries of the inverse of the matrix $\big(g'_{\mathfrak{s},\mathfrak{t}}\big)_{\mathfrak{s},\mathfrak{t}\in\mathbb{T}^r(\lambda)}$. A comparison with [7, Prop. 3.3] shows that

$$\tilde{g}_{\mathfrak{s},\mathfrak{t}} = p^*_{d(\mathfrak{s})\sigma_\lambda, d(\mathfrak{t})\sigma_\lambda}\in v^{-1}\mathbb{Z}[v^{-1}] \qquad \text{if } \mathfrak{s}\neq\mathfrak{t}.$$

Hence we also have $g'_{\mathfrak{s},\mathfrak{t}}\in v^{-1}\mathbb{Z}[v^{-1}]$ for $\mathfrak{s}\neq\mathfrak{t}$. □

Remark 3.7. Let $\lambda\in\Lambda_n$ and $\mathfrak{C}$ be any left cell such that $\mathfrak{C}$ and $\mathfrak{C}_\lambda$ are contained in the same two-sided cell. Then, by Theorem 2.4, $[\mathfrak{C}]_A$ and $[\mathfrak{C}_\lambda]_A$ are canonically isomorphic as $\mathcal{H}_n$-modules. Hence, in combination with Theorem 3.6, we conclude that $\tilde{S}^\lambda\cong[\mathfrak{C}]_A$. Thus, any left cell module of $\mathcal{H}_n$ is canonically isomorphic to a Specht module.

Remark 3.8. The above results also hold for specialized algebras. More precisely, let R be any commutative ring (with 1) and fix two invertible elements $Q,q\in R$ which admit square roots $Q^{1/2}$ and $q^{1/2}$ in R. Then we have a unique ring homomorphism $\theta\colon A\to R$ such that $\theta(V)=Q^{1/2}$ and $\theta(v)=q^{1/2}$. We can extend scalars from A to R and set

$$\mathcal{H}_{n,R} = R\otimes_A \mathcal{H}_n, \quad \tilde{S}^\lambda_R = R\otimes_A \tilde{S}^\lambda, \quad [\mathfrak{C}]_R = R\otimes_A [\mathfrak{C}]_A,$$

for any $\lambda \in \Lambda_n$ and any left cell $\mathfrak{C}$ of W_n. Then $\tilde{S}_R^\lambda$ precisely is the Specht module of the algebra of type B_n with parameters Q, q, as considered by Dipper–James–Murphy [6]. By Theorem 3.6 and Remark 3.7, we have an induced canonical isomorphism $\tilde{\varphi}_R \colon \tilde{S}_R^\lambda \xrightarrow{\sim} [\mathfrak{C}]_R$ whenever $\mathfrak{C}$ is in the same two-sided cell as $\mathfrak{C}_\lambda$.

4. Counterexample

Recall that $\mathcal{H}_n$ is defined over the ring of Laurent polynomials $A = \mathbb{Z}[V^{\pm 1}, v^{\pm 1}]$ in two independent indeterminates. In the previous sections, we considered the Kazhdan–Lusztig cell modules of $\mathcal{H}_n$ with respect to the "asymptotic case" [3], that is, assuming that the group of monomials $\{V^i v^j \mid i, j \in \mathbb{Z}\}$ is endowed with the pure lexicographic order where $V^i v^j < v^{i'} v^{j'}$ if and only if $i < i'$ or $i = i'$ and $j < j'$. But there are many other monomial orders, each giving rise to a Kazhdan–Lusztig basis of $\mathcal{H}_n$ and corresponding cell modules.

The aim of this section is to show that, in general, the Dipper–James–Murphy Specht modules $\tilde{S}^\lambda$ cannot be identified with cell modules for these other choices of a monomial order. We do this in two ways: (1) by a concrete example in type B_3 and (2) by a general argument involving non-semisimple specialisations of $\mathcal{H}_n$.

4.A. An example in type B_3.

Let $n = 3$; then $W_3 = \langle t, s_1, s_2 \rangle$. Let $\lambda = ((1), (2)) \in \Lambda_3$ and consider the corresponding Specht module $\tilde{S}^\lambda$. By Theorem 3.6, it is isomorphic to $[\mathfrak{C}_\lambda]_A$, where $\mathfrak{C}_\lambda$ is a left cell with respect to the "asymptotic case". We have $l = 1$ and $\sigma_\lambda a_l = s_2 t$. Using Proposition 2.6, we find that

$$\mathfrak{C}_\lambda = \{s_2 t, \quad s_1 s_2 t, \quad s_2 s_1 s_2 t\}.$$

The corresponding left cell representation $\rho_\lambda \colon \mathcal{H}_3 \to M_3(A)$ is given by

$$T_t \mapsto \begin{pmatrix} V & Vv^{-1}V^{-1}v & Vv^{-2} + V^{-1}v^2 \\ 0 & -V^{-1} & 0 \\ 0 & 0 & -V^{-1} \end{pmatrix},$$

$$T_{s_1} \mapsto \begin{pmatrix} -v^{-1} & 0 & 0 \\ 1 & v & 0 \\ 0 & 0 & v \end{pmatrix}, \qquad T_{s_2} \mapsto \begin{pmatrix} v & 1 & 0 \\ 0 & -v^{-1} & 0 \\ 0 & 1 & v \end{pmatrix}.$$

Now let us choose a different monomial order on $\{V^i v^j \mid i, j \in \mathbb{Z}\}$, namely, the weighted lexicographic order where

$$V^i v^j < V^{i'} v^{j'} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad i + j < i' + j' \quad \text{or} \quad i + j = i' + j' \text{ and } i < i'.$$

(In particular, we have $v < V < v^2$.) By an explicit computation, one can show that, in this case, the Kazhdan–Lusztig cell modules are all irreducible over K (the field of fractions of A), in accordance with [2, Conjecture A$^+$]. (We are in the case $r = 1$ of that conjecture.) Furthermore, there are precisely three left cells $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ such that $[\mathfrak{C}_i]_K \cong \tilde{S}_K^\lambda$; they are given as follows:

$$\begin{aligned} \mathfrak{C}_1 &= \{s_1 s_2 s_1, \quad s_1 t s_1 s_2 s_1, \quad t s_1 s_2 s_1\}, \\ \mathfrak{C}_2 &= \{s_1 s_2 s_1 t, \quad s_1 t s_1 s_2 s_1 t, \quad t s_1 s_2 s_1 t\}, \\ \mathfrak{C}_3 &= \{s_1 s_2 s_1 t s_1, \quad s_1 t s_1 s_2 s_1 t s_1, \quad t s_1 s_2 s_1 t s_1\}. \end{aligned}$$

The corresponding left cell representations are all identical and given by:

$$T_t \mapsto \begin{pmatrix} -V^{-1} & 0 & 0 \\ 0 & -V^{-1} & 0 \\ 1 & Vv^{-1}+V^{-1}v & V \end{pmatrix},$$

$$T_{s_1} \mapsto \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & 0 & -v^{-1} \end{pmatrix}, \quad T_{s_2} \mapsto \begin{pmatrix} v & Vv^{-2}+V^{-1}v^2 & 0 \\ 0 & -v^{-1} & 0 \\ 0 & 1 & v \end{pmatrix}.$$

Denote this representation by $\rho\colon \mathcal{H}_3 \to M_3(A)$. Now one checks that $P\rho_\lambda(T_s) = \rho(T_s)P$ for $s \in \{t, s_1, s_2\}$ where

$$P = \begin{pmatrix} 0 & 0 & Vv^{-2}+V^{-1}v^2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus, P defines a non-trivial module homomorphism between ρ_λ and ρ. Since these representations are irreducible over K, the matrix P is uniquely determined up to scalar multiples. But we see that there is no scalar $\lambda \in K$ such that $\lambda P \in M_3(A)$ and $\det(\lambda P) \in A^\times$. Hence, $\tilde{S}^\lambda$ will not be isomorphic to any Kazhdan–Lusztig cell module with respect to the above weighted lexicographic order.

4.B. **General cell modules.**

Let k be a field and fix an element $\xi \in k^\times$. Let $a, b \in \mathbb{Z}_{\geqslant 0}$ and consider the specialisation $A \to k$ such that $V \mapsto \xi^b$ and $v \mapsto \xi^a$. Let $\mathcal{H}_{n,k} = k \otimes_A \mathcal{H}_n$ be the corresponding specialized algebra. As in Remark 3.8, we also have corresponding Specht modules $\tilde{S}_k^\lambda$ for $\mathcal{H}_{n,k}$. Now, for each $\lambda \in \Lambda_n$, there is a certain $\mathcal{H}_{n,k}$-invariant bilinear form $\phi_\lambda\colon \tilde{S}_k^\lambda \times \tilde{S}_k^\lambda \to k$; see [6, §5]. Let $\mathrm{rad}(\phi_\lambda)$ be the radical of that form and set $D^\lambda = \tilde{S}_k^\lambda/\mathrm{rad}(\phi_\lambda)$. Then D^λ is either 0 or an absolutely irreducible $\mathcal{H}_{n,k}$-module; furthermore, we have

$$\mathrm{Irr}(\mathcal{H}_{n,k}) = \{D^\mu \mid \mu \in \Lambda^\clubsuit\} \quad \text{where} \quad \Lambda^\clubsuit = \{\lambda \in \Lambda \mid D^\lambda \neq 0\};$$

see Dipper–James–Murphy [6, Theorem 6.6]. The conjecture in [6, 8.13] about an explicit combinatorial description of $\Lambda^\clubsuit$ has recently been proved by Ariki–Jacon [1].

Now consider the Kazhdan–Lusztig basis $\{C_w\}$ of $\mathcal{H}_{n,k}$ with respect to the weight function $L\colon W_n \to \mathbb{Z}$ such that $L(t) = b$ and $L(s_i) = a$ for all i. Assume that Lusztig's conjectures **(P1)**–**(P15)** in [14, 14.2] on Hecke algebras with unequal parameters hold. (This is the case, for example, in the "equal parameter case" where $a = b$; see [14, Chap. 15].) Using these properties, it is shown in [9] that $\mathcal{H}_{n,k}$ has a natural "cellular structure" in the sense of Graham–Lehrer [12]. The elements of the "cellular basis" are certain linear combinations of the basis elements $\{C_w\}$. Then, by the general theory of "cellular algebras", for any $\lambda \in \Lambda_n$, we have a "cell module" $W_k(\lambda)$ for $\mathcal{H}_{n,k}$ and this cell module is naturally equipped with an $\mathcal{H}_{n,k}$-invariant bilinear form $g_\lambda\colon W_k(\lambda) \times W_k(\lambda) \to k$. Let $\mathrm{rad}(g_\lambda)$ be the radical of that form and set $L^\lambda = W_k(\lambda)/\mathrm{rad}(g_\lambda)$. Then, again, L^λ is either 0 or an absolutely irreducible $\mathcal{H}_{n,k}$-module; furthermore, we have

$$\mathrm{Irr}(\mathcal{H}_{n,k}) = \{L^\mu \mid \mu \in \Lambda^\spadesuit\} \quad \text{where} \quad \Lambda^\spadesuit = \{\lambda \in \Lambda \mid L^\lambda \neq 0\};$$

see Graham–Lehrer [12, §3] and [9, Example 4.4].

Given these two settings, it is natural to ask if $\tilde{S}_k^\lambda \cong W_k(\lambda)$ and, subsequently, if $\Lambda^\clubsuit = \Lambda^\spadesuit$? In the case where $\mathcal{H}_{n,k}$ is semisimple, it is shown in [9, Example 4.4] that $\tilde{S}_k^\lambda \cong W_k(\lambda)$; furthermore, by the general theory of "cellular algebras" [12] and the results in [6], we have $\Lambda^\clubsuit = \Lambda^\spadesuit = \Lambda$ in

this case. However, if $\mathcal{H}_{n,k}$ is not semisimple, then the answer to these questions is negative, as can be seen from the fact that $\Lambda^{\clubsuit} \neq \Lambda^{\spadesuit}$ in general; see [11] and the references there.

By [11, Theorem 2.8] it is true, however, that $\Lambda^{\clubsuit} = \Lambda^{\spadesuit}$ if $b > (n-1)a > 0$ which corresponds precisely to the "asymptotic case" discussed in this paper. Indeed, by [8, Corollary 6.3], the basis $\{C_w\}$ of $\mathcal{H}_{n,k}$ is cellular under this assumption on a, b, and by Theorem 3.6, we have $W_k(\lambda) \cong \tilde{S}_k^\lambda$.

Acknowledgements. The final form of this paper grew out of several discussions which the three authors could hold thanks to the hospitality of various institutions. Part of this work was done while all three authors enjoyed the hospitality of the Bernoulli Center at the EPFL Lausanne (Switzerland) in 2005. C.P. thanks the University of Aberdeen (Scotland) for an invitation in November 2006; M.G. and L.I. would like to thank the University of Cyprus at Nicosia for an invitation in March 2007.

References

[1] S. Ariki and N. Jacon, Dipper–James–Murphy's conjecture for Hecke algebras of type B, preprint; available at `math.RT/0703447`.

[2] C. Bonnafé, M. Geck, L. Iancu and T. Lam, On domino insertion and Kazhdan–Lusztig cells in type B_n, preprint; available at `math.RT/0609279`.

[3] C. Bonnafé and L. Iancu, Left cells in type B_n with unequal parameters, Represent. Theory **7** (2003), 587–609.

[4] C. Bonnafé, Two-sided cells in type B in the asymptotic case, J. Algebra **304** (2006), 216–236.

[5] R. Dipper and G. D. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. **52** (1986), 20–52.

[6] R. Dipper, G. D. James and G. E. Murphy, Hecke algebras of type B_n at roots of unity, Proc. London Math. Soc. **70** (1995), 505–528.

[7] M. Geck, On the induction of Kazhdan–Lusztig cells, Bull. London Math. Soc. **35** (2003), 608–614.

[8] M. Geck, Relative Kazhdan–Lusztig cells, Represent. Theory **10** (2006), 481–524.

[9] M. Geck, Hecke algebras of finite type are cellular, Invent. Math. (2007), to appear.

[10] M. Geck and L. Iancu, Lusztig's a-function in type B_n in the asymptotic case. Special issue celebrating the 60th birthday of George Lusztig, Nagoya J. Math. **182** (2006), 199–240.

[11] M. Geck and N. Jacon, Canonical basic sets in type B. Special issue in honour of Gordon Douglas James, J. Algebra **306** (2006), 104–127.

[12] J. J. Graham and G. I. Lehrer, Cellular algebras, Invent. Math. **123** (1996), 1–34.

[13] G. Lusztig, Left cells in Weyl groups, Lie Group Representations, I (R. L. R. Herb and J. Rosenberg, eds.), Lecture Notes in Math., vol. 1024, Springer-Verlag, 1983, pp. 99–111.

[14] G. Lusztig, Hecke algebras with unequal parameters, CRM Monographs Ser. **18**, Amer. Math. Soc., Providence, RI, 2003.

[15] T. P. McDonough and C. A. Pallikaros, On relations between the classical and the Kazhdan–Lusztig representations of symmetric groups and associated Hecke algebras, J. Pure and Applied Algebra **203** (2005), 133–144.

M.G. and L.I.: Department of Mathematical Sciences, King's College, Aberdeen University, Aberdeen AB24 3UE, Scotland, U.K.

E-mail address: `m.geck@maths.abdn.ac.uk`

E-mail address: `l.iancu@maths.abdn.ac.uk`

C.P.: Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

E-mail address: `pallikar@ucy.ac.cy`